Journal of Mathematical Sciences: Advances and Applications
Volume 3, Number 2, 2009, Pages 189-198

SOME FIXED POINT THEOREMS OF ORDERED CONTRACTIVE MAPPINGS

XIAN ZHANG

School of Sciences
Jimei University
Xiamen 361021
P. R. China
e-mail: zhangxianjmj@sina.com

Abstract

In this paper we introduce some types of ordered contractive mappings and obtain some fixed point theorems of nonlinear mappings in ordered Banach spaces.

1. Introduction

Let E be a real Banach space. $P \subset E$ is called a *cone* in E if : (i) P is nonempty convex closed; (ii) for any $x \in P$ and any real $\lambda > 0$, $\lambda x \in P$; (iii) $x \in P$ and $-x \in P$ implies $x = \theta$. For a cone P in E, we can define the ordering "$\leq_P$" in $E : x \leq_P y$, if and only if $y - x \in P$ for all x, y in E. A cone P in E is called *normal*, if there exists a constant number $N > 0$, such that $\theta \leq_P x \leq_P y$ implies $\|x\| \leq N\|y\|$, N is called *normal constant* of P.

The fixed point theorems of nonlinear mappings in ordered Banach spaces have been studied by many authors. In this paper, we introduce some types of ordered contractive mappings and obtain some fixed point theorems.

2000 Mathematics Subject Classification: 47H10.

Keywords and phrases: fixed point, ordered contractive mapping, ordered Banach space.

Received February 19, 2009

Let "$\leq_P$" be ordering defined by cone P, for $u, v \in E$, if either $u \leq_P v$ or $v \leq_P u$ holds, we said that u and v are P-comparable. If u and v are P-comparable, we set $u \vee_P v = u$, if $u \geq_P v$ and $u \vee_P v = v$ if $u \leq_P v$.

Lemma 1. *Let P be a cone in E. If u and v are P-comparable, then $u - v$ and $v - u$ are P-comparable and*

$$\theta \leq_P (u - v) \vee_P (v - u).$$

Lemma 2. *Let P be a cone in E and let S be a bounded linear operator from E to E with bounded inverse S^{-1}. Then*

(1) *$S(P)$ is a cone in E.*

(2) *For every $x, y \in E$, $x \leq_P y$, if and only if $S(x) \leq_{S(P)} S(y)$.*

(3) *If P is a normal cone with normal constant N, then $S(P)$ is a normal cone in E with normal constant $\|S\|N\|S^{-1}\|$.*

(4) *For every $x, y \in E$, x and y are P-comparable, if and only if $S(x)$ and $S(y)$ are $S(P)$-comparable and*

$$S(x \vee_P y) = S(x) \vee_{S(P)} S(y).$$

In this paper, we always suppose that Λ is a family of normal cone with common normal constant N and for every $P, Q \in \Lambda$, there exists a bounded linear operator $S_{PQ} : E \to E$ with bounded inverse S_{PQ}^{-1} such that $S(P) = Q$ and $\|S_{PQ}\| = \|S_{PQ}^{-1}\| = 1$. For every $P \in \Lambda$, $S_{PP} = I$ (the identity operator on E).

For $a, b \in E$, a and b are called Λ-*comparable*, if there exists $P \in \Lambda$ such that a and b are P-comparable. The sequence $\{x_n\}$ is called Λ-*comparable*, if for all n, m, x_n and x_m are Λ-comparable. Λ is said to have convergent sequence comparable property, if for every Λ-comparable convergence sequence $\{x_n\}$ with limit x^*, x_n and x^* are Λ-comparable for all n.

2. Fixed Point Theorems

Theorem 1. *Let mapping* $T : E \to E$ *is continuous and* $0 < \lambda < 1$ *satisfies*

(i) *If* a *and* b *are* $P \in \Lambda$ *comparable, then there exists* $Q \in \Lambda$ *such that* Ta *and* Tb *are* Q*-comparable and*

(ii) $(Ta - Tb) \vee_Q (Tb - Ta) \leq_Q \lambda S_{PQ}((a - b) \vee_P (b - a))$;

(iii) *there exists* a $x_0 \in E$, *such that* x_0 *and* Tx_0 *are* Λ*-comparable.*

Then T *has fixed point and the iterative sequence* $\{T^n x_0\}$ *converges to a fixed point* x^* *of* T*. Moreover* $\|x^* - x_0\| \leq (1 + \frac{N\lambda}{1 - \lambda})\|Tx_0 - x_0\|$.

Proof. Considering the iterative sequence

$$x_1 = Tx_0,\ x_2 = Tx_1 = T^2 x_0,\ \cdots,\ x_{n+1} = Tx_n = T^{n+1} x_0,\ \cdots.$$

By conditions (i)-(iii), there exists $P_1 \in \Lambda$ such that x_0 and $x_1 = Tx_0$ are P_1-comparable. There exists $P_2 \in \Lambda$ such that x_1 and $x_2 = Tx_1$ are P_2-comparable and

$$\theta \leq_{P_2} (x_1 - x_2) \vee_{P_2} (x_2 - x_1) = (Tx_0 - Tx_1) \vee_{P_2} (Tx_1 - Tx_0)$$

$$\leq_{P_2} \lambda S_{P_1 P_2}((x_0 - x_1) \vee_{P_1} (x_1 - x_0)).$$

Also there exists $P_3 \in \Lambda$ such that x_2 and $x_3 = Tx_2$ are P_3-comparable and

$$\theta \leq_{P_3} (x_2 - x_3) \vee_{P_3} (x_3 - x_2) = (Tx_1 - Tx_2) \vee_{P_3} (Tx_2 - Tx_1)$$

$$\leq_{P_3} \lambda S_{P_2 P_3}((x_1 - x_2) \vee_{P_2} (x_2 - x_1)).$$

$$\leq_{P_3} \lambda^2 S_{P_2 P_3} S_{P_1 P_2}((x_0 - x_1) \vee_{P_1} (x_1 - x_0)).$$

Generally there exists $P_{n+1} \in \Lambda$ such that x_n and $x_{n+1} = Tx_n$ are P_{n+1}-comparable and

$$\begin{aligned}
\theta &\leq_{P_{n+1}} (x_n - x_{n+1}) \vee_{P_{n+1}} (x_{n+1} - x_n) \\
&= (Tx_{n-1} - Tx_n) \vee_{P_{n+1}} (Tx_n - Tx_{n-1}) \\
&\leq_{P_{n+1}} \lambda S_{P_n P_{n+1}} ((x_{n-1} - x_n) \vee_{P_n} (x_n - x_{n-1})) \\
&\leq_{P_{n+1}} \lambda^2 S_{P_n P_{n+1}} S_{P_{n-1} P_n} ((x_{n-2} - x_{n-1}) \vee_{P_{n-1}} (x_{n-1} - x_{n-2})) \\
&\leq \cdots\cdots \\
&\leq_{P_{n+1}} \lambda^n S_{P_n P_{n+1}} S_{P_{n-1} P_n} \cdots S_{P_1 P_2} ((x_0 - x_1) \vee_{P_1} (x_1 - x_0)).
\end{aligned}$$

By the normality of P_{n+1},

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \lambda^n N \|S_{P_n P_{n+1}} S_{P_{n-1} P_n} \cdots S_{P_1 P_2}\| \|(x_0 - x_1) \vee_{P_1} (x_1 - x_0)\| \\
&\leq \lambda^n N \|x_0 - x_1\|.
\end{aligned}$$

Since $\lambda < 1$, it is easy to prove that $\{x_n\}$ is a Cauchy sequence. Since E is complete, let $x_n \to x^* \in E$. The continuity of T implies that

$$Tx^* = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*,$$

so x^* is a fixed point of T. Moreover

$$\begin{aligned}
\|x^* - x_0\| = \lim_{n\to\infty} \|x_n - x_0\| &\leq \lim_{n\to\infty} \sum_{i=1}^{n} \|x_i - x_{i-1}\| \\
&\leq \lim_{n\to\infty} \sum_{i=2}^{n} \lambda^{i-1} N \|x_1 - x_0\| + \|x_1 - x_0\| \\
&\leq \sum_{n=2}^{\infty} \lambda^{n-1} N \|x_1 - x_0\| + \|x_1 - x_0\| \\
&= \left(1 + \frac{N\lambda}{1-\lambda}\right) \|x_1 - x_0\|.
\end{aligned}$$

The proof is complete.

Theorem 2. *Let mapping* $T : E \to E$ *and* $0 < \lambda < 1$ *satisfies*

(i) *If* a *and* b *are* $P \in \Lambda$ *comparable, then there exists* $Q \in \Lambda$ *such that* Ta *and* Tb *are* Q*-comparable and*

(ii) $(Ta - Tb) \vee_Q (Tb - Ta) \leq_Q \lambda S_{PQ}((a - b) \vee_P (b - a))$;

(iii) *there exists a* $x_0 \in E$, *such that for all* n, x_0 *and* $T^n x_0$ *are* Λ*-comparable*;

(iv) Λ *has convergent sequence comparable property*.

Then T *has fixed point and the iterative sequence* $\{T^n x_0\}$ *converges to a fixed point* x^* *of* T.

Proof. As the proof of Theorem 1, we define iterative sequence $x_n = T^n x_0$ and we can prove that $\{x_n\}$ is a Cauchy sequence. Since E is complete, let $x_n \to x^* \in E$. Next we prove that x^* is a fixed point of T.

Now we prove that for all n and m, x_n and x_m are Λ-comparable. Let $m < n$, by the condition (iii) x_0 and $x_{n-m} = T^{n-m} x_0$ are Λ-comparable. And by the condition (i), we get that Tx_0 and Tx_{n-m} are Λ-comparable. Repeating the procedure, we have $T^m x_0$ and $T^m x_{n-m} = T^n x_0$ are Λ-comparable. So x_n and x_m are Λ-comparable.

Since Λ has convergent sequence comparable property, for all n, x_n and x^* are Λ-comparable. Hence there exist $P_n \in \Lambda$ such that x_n and x^* are P_n-comparable. By condition (i) and (ii), there exist $Q_n \in \Lambda$ such that Tx^* and Tx_n are Q_n-comparable and

$$\theta \leq_{Q_n} (Tx^* - Tx_n) \vee_{Q_n} (Tx_n - Tx^*) \leq_{Q_n} \lambda S_{P_n Q_n}((x^* - x_n) \vee_{P_n} (x_n - x^*)).$$

By the normality of Q_n,

$$\|x_{n+1} - Tx^*\| = \|Tx_n - Tx^*\| \leq \lambda N \|x_n - x^*\| \to 0, \ (n \to \infty).$$

This implies $x_{n+1} \to Tx^*$. We get $Tx^* = x^*$ and x^* is a fixed point of T.

Theorem 3. *Let mapping* $T : E \to E$ *is continuous and* $0 < \lambda N < \frac{1}{2}$ *satisfies*

(i) *If* a *and* b *are* $P \in \Lambda$ *comparable and* b *and* c *are also* P-*comparable, then there exists* $Q \in \Lambda$ *such that* Ta *and* Tb *are* Q-*comparable and* Tb *and* Tc *are also* Q-*comparable.*

(ii) *If* a *and* Ta *are* $P \in \Lambda$ *comparable and* b *and* Tb *are also* P-*comparable, then there exists* $Q \in \Lambda$ *such that* Ta *and* Tb *are* Q-*comparable and*

$$(Ta - Tb) \vee_Q (Tb - Ta)$$
$$\leq_Q \lambda S_{PQ}((Ta - a) \vee_P (a - Ta) + (Tb - b) \vee_P (b - Tb));$$

(iii) *there exists a* $x_0 \in E$ *and* $P_1 \in \Lambda$, *such that* x_0 *and* Tx_0 *are* P_1- *comparable and* Tx_0 *and* $T^2 x_0$ *are* P_1-*comparable.*

Then T has fixed point and the iterative sequence $\{T^n x_0\}$ *converges to a fixed point* x^* *of T.*

Proof. Defining the iterative sequence

$$x_1 = Tx_0,\ x_2 = Tx_1 = T^2 x_0,\ \cdots,\ x_{n+1} = Tx_n = T^{n+1} x_0,\ \cdots.$$

Since x_0 and $x_1 = Tx_0$ are P_1-comparable and x_1 and $x_2 = Tx_1$ are P_1-comparable. By condition (ii), there exists Q_1 such that Tx_0 and Tx_1 are Q_1-comparable and

$$\theta \leq_{Q_1} (x_1 - x_2) \vee_{Q_1} (x_2 - x_1) = (Tx_0 - Tx_1) \vee_{Q_1} (Tx_1 - Tx_0)$$
$$\leq_{Q_1} \lambda S_{P_1 Q_1}(((x_0 - Tx_0) \vee_{P_1} (Tx_0 - x_0)) + ((x_1 - Tx_1) \vee_{P_1} (Tx_1 - x_1))).$$

Hence

$$\|x_1 - x_2\| \leq \lambda N(\|x_0 - x_1\| + \|x_1 - x_2\|),$$

$$\|x_1 - x_2\| \leq \frac{\lambda N}{1 - \lambda N} \|x_0 - x_1\|.$$

By condition (i), there exists $P_2 \in \Lambda$ such that x_1 and $x_2 = Tx_1$ are P_2-comparable and x_2 and $x_3 = Tx_2$ are P_2-comparable. By condition (ii), there exists Q_2 such that Tx_1 and Tx_2 are Q_2-comparable and

$$\theta \leq_{Q_2} (x_2 - x_3) \vee_{Q_2} (x_3 - x_2) = (Tx_1 - Tx_2) \vee_{Q_2} (Tx_2 - Tx_1)$$

$$\leq_{Q_2} \lambda S_{P_2 Q_2}(((x_1 - Tx_1) \vee_{P_2} (Tx_1 - x_1)) + ((x_2 - Tx_2) \vee_{P_2} (Tx_2 - x_2))).$$

Hence

$$\|x_2 - x_3\| \leq \lambda N(\|x_1 - x_2\| + \|x_2 - x_3\|),$$

$$\|x_2 - x_3\| \leq \frac{\lambda N}{1 - \lambda N} \|x_1 - x_2\|.$$

Generally we have

$$\|x_n - x_{n+1}\| \leq \frac{\lambda N}{1 - \lambda N} \|x_{n-1} - x_n\|.$$

Since $\lambda N < \frac{1}{2}$, we can prove that $\{x_n\}$ is a Cauchy sequence in E. Since E is complete, let $x_n \to x^* \in E$. The continuity of T implies that

$$Tx^* = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*,$$

so x^* is a fixed point of T. The proof is complete.

Theorem 4. *Let mapping* $T : E \to E$ *is continuous and* $0 < \lambda N < 1/2$ *satisfies*

(i) *If* a *and* b *are* $P \in \Lambda$ *comparable*, *then there exists* $Q \in \Lambda$ *such that* Ta *and* Tb *are* Q-*comparable*

(ii) *If* a *and* Ta *are* $P \in \Lambda$ *comparable*, b *and* Tb *are* $Q \in \Lambda$ *comparable and* Ta *and* Tb *are* $R \in \Lambda$ *comparable*, *then*

$$(Ta - Tb) \vee_R (Tb - Ta)$$

$$\leq_R \lambda(S_{PR}(Ta - a) \vee_P (a - Ta)) + S_{QR}((Tb - b) \vee_P (b - Tb));$$

(iii) *there exists a* $x_0 \in E$ *and* $P_1 \in \Lambda$, *such that* x_0 *and* Tx_0 *are* P_1*-comparable. Then* T *has fixed point and the iterative sequence* $\{T^n x_0\}$ *converges to a fixed point* x^* *of* T.

Proof. Defining the iterative sequence

$$x_1 = Tx_0,\ x_2 = Tx_1 = T^2 x_0,\ \cdots,\ x_{n+1} = Tx_n = T^{n+1}x_0,\ \cdots.$$

By conditions, there exists $P_n \in \Lambda$ such that x_{n-1} and x_n are P_n-comparable. And

$$\begin{aligned}\theta &\leq_{P_{n+1}} (x_n - x_{n+1}) \vee_{P_{n+1}} (x_{n+1} - x_n)\\ &= (Tx_{n-1} - Tx_n) \vee_{P_{n+1}} (Tx_n - Tx_{n-1})\\ &\leq_{P_{n+1}} \lambda(S_{P_n P_{n+1}}((x_{n-1} - x_n) \vee_{P_n} (x_n - x_{n-1}))\\ &\quad + S_{P_{n+1}P_{n+1}}((x_n - x_{n+1}) \vee_{P_{n+1}} (x_{n+1} - x_n)).\end{aligned}$$

So

$$\|x_n - x_{n+1}\| \leq \frac{\lambda N}{1 - \lambda N}\|x_{n-1} - x_n\|.$$

It is easy to prove that $\{x_n\}$ is a Cauchy sequence and converges to $x^* \in E$. By the continuity of T, $Tx^* = x^*$, so x^* is a fixed point of T. The proof is complete.

Theorem 5. *Let mapping* $T : E \to E$ *is continuous and* $0 < \lambda N < 1/2$ *satisfies*

(i) *If* a *and* b *are* $P \in \Lambda$ *comparable, then there exists* $Q \in \Lambda$ *such that* Ta *and* Tb *are* Q*-comparable.*

(ii) *If* a *and* Tb *are* $P \in \Lambda$ *comparable,* b *and* Ta *are also* Q*-comparable and* Ta *and* Tb *are* $R \in \Lambda$ *comparable, then*

$$\begin{aligned}&(Ta - Tb) \vee_R (Tb - Ta)\\ &\leq_R \lambda(S_{PR}((a - Tb) \vee_P (Tb - a)) + S_{QR}((b - Ta) \vee_P (Ta - b)));\end{aligned}$$

(iii) *there exists a* $x_0 \in E$ *such that* x_0 *and* Tx_0 *are* Λ*-comparable and* x_0 *and* T^2x_0 *are* Λ*-comparable.*

Then T has fixed point and the iterative sequence $\{T^n x_0\}$ *converges to a fixed point* x^* *of T.*

Proof. Defining the iterative sequence

$$x_1 = Tx_0,\ x_2 = Tx_1 = T^2x_0,\ \cdots,\ x_{n+1} = Tx_n = T^{n+1}x_0,\ \cdots.$$

By condition (iii), there exists $P_1 \in \Lambda$ such that x_0 and $x_1 = Tx_0$ are P_1-comparable and there exists $Q_1 \in \Lambda$ such that x_0 and $x_2 = T^2x_0$ are Q_1-comparable. By condition (i), there exists $P_2 \in \Lambda$ such that x_1 and $x_2 = Tx_1$ are P_2-comparable and there exists $Q_2 \in \Lambda$ such that x_1 and $x_3 = T^2x_1$ are Q_2-comparable. Generally there exists $P_n \in \Lambda$ such that x_{n-1} and $x_n = Tx_{n-1}$ are P_n-comparable and there exists $Q_n \in \Lambda$ such that x_{n-1} and $x_{n+1} = T^2x_{n-1}$ are Q_n-comparable.

Since x_{n-1} and Tx_n are Q_n-comparable, x_n and Tx_{n-1} are also Q_n-comparable and Tx_{n-1} and Tx_n are P_{n+1}-comparable, by condition (ii), we have

$$\begin{aligned}
\theta &\leq_{P_{n+1}} (x_n - x_{n+1}) \vee_{P_{n+1}} (x_{n+1} - x_n) \\
&= (Tx_{n-1} - Tx_n) \vee_{P_{n+1}} (Tx_n - Tx_{n-1}) \\
&\leq_{P_{n+1}} \lambda S_{P_{n+1}Q_n}((x_{n-1} - Tx_n) \vee_{Q_n} (Tx_n - x_{n-1}) \\
&\quad +(x_n - Tx_{n-1}) \vee_{Q_n} (Tx_{n-1} - x_n)).
\end{aligned}$$

So

$$\|x_n - x_{n+1}\| \leq \lambda N \|x_{n-1} - x_{n+1}\| \leq \lambda N (\|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|),$$

$$\|x_n - x_{n+1}\| \leq \frac{\lambda N}{1 - \lambda N} \|x_{n-1} - x_n\|.$$

Since $\lambda N < 1/2$, it is easy to prove that $\{x_n\}$ is a Cauchy sequence and converges to $x^* \in E$. By the continuity of T, $Tx^* = x^*$, so x^* is a fixed point of T. The proof is complete.

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